Differential Geometry IV: General Relativity

5.1 Let $\mathcal{M}$ be a differentiable manifold and ∇ a connection on $\mathcal{M}$.

(a) Show that there exists no $(1,2)$-type tensor field A on $\mathcal{M}$ with the property that, in any local coordinate system $(x^1, \ldots, x^n)$ on $\mathcal{M}$

$$A_{ij}^k = \Gamma_{ij}^k.$$

Hint: Check how Γ_{ij}^k transforms under changes of coordinates.

(b) Show that the torsion $T : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ of the connection ∇, which is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

is a tensor field.

(c) Let $\bar{\nabla}$ be a (possibly) different connection on $\mathcal{M}$. Show that the difference $\nabla - \bar{\nabla} : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ is also a tensor field. Deduce that, there exists a $(1,2)$-type tensor field A such that, in any given local coordinate system $(x^1, \ldots, x^n)$,

$$A_{ij}^k = \Gamma_{ij}^k - \bar{\Gamma}_{ij}^k$$

where Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$ are the Christoffel symbols of ∇ and $\bar{\nabla}$, respectively.

Solution. (a) Assume that there exists a tensor field A as in the statement. Then, if $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ are two coordinate systems around the same point $p \in \mathcal{M}$, the components A_{ij}^k and $\tilde{A}_{ij}^k$ of A in the two coordinate systems, respectively, are related by the transformation formula

$$\tilde{A}_{ij}^k = A_{\alpha\beta}^\gamma \cdot \frac{\partial y^k}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^i} \frac{\partial x^\beta}{\partial y^j}. \tag{1}$$

On the other hand, the Christoffel symbols Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ in the coordinate systems $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$, respectively, are given by the relations

$$\Gamma_{ij}^k = dx^k \Big(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \Big)$$

and

$$\begin{aligned}
\tilde{\Gamma}_{ij}^k &= dy^k \Big(\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} \Big) \\
&= \frac{\partial y^k}{\partial x^\gamma} dx^\gamma \Big(\nabla_{\frac{\partial x^\alpha}{\partial y^i} \cdot \frac{\partial}{\partial x^\alpha}} \big(\frac{\partial x^\beta}{\partial y^j} \cdot \frac{\partial}{\partial x^\beta} \Big) \\
&= \frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot dx^\gamma \Big(\nabla_{\frac{\partial}{\partial x^\alpha}} \big(\frac{\partial x^\beta}{\partial y^j} \cdot \frac{\partial}{\partial x^\beta} \Big) \\
&= \frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot dx^\gamma \Big(\frac{\partial}{\partial x^\alpha} \big(\frac{\partial x^\beta}{\partial y^j} \big) \cdot \frac{\partial}{\partial x^\beta} + \frac{\partial x^\beta}{\partial y^j} \cdot \nabla_{\frac{\partial}{\partial x^\alpha}} \big(\frac{\partial}{\partial x^\beta} \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot\Big(\frac{\partial}{\partial x^\alpha}\Big(\frac{\partial x^\beta}{\partial y^j}\Big)\cdot dx^\gamma\big(\frac{\partial}{\partial x^\beta}\big)+\frac{\partial x^\beta}{\partial y^j}\cdot dx^\gamma\big(\nabla_{\frac{\partial}{\partial x^\alpha}}\big(\frac{\partial}{\partial x^\beta}\big)\big)\Big)\\
&= \frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot\frac{\partial}{\partial x^\alpha}\Big(\frac{\partial x^\gamma}{\partial y^j}\Big)+\Gamma^\gamma_{\alpha\beta}\cdot\frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot\frac{\partial x^\beta}{\partial y^j}
\end{aligned}
$$

(note that we used the fact that that $dx^k(\cdot)$ is a tensor field and, thus, is $C^\infty(\mathcal{M})$-linear in its argument). Therefore, we see that the transformation law for the Christoffel symbols contains an additional term which is not there in (1), namely $\frac{\partial y^k}{\partial x^\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\cdot\frac{\partial}{\partial x^\alpha}$. Expressing the coordinates $y^i=y^i(x)$ as functions of $(x^1,\dots,x^n)$, this term is equal to

$$
[Dy]^k_\gamma\cdot\big([Dy]^{-1}\big)^\alpha_i\cdot\Big(\frac{\partial}{\partial x^\alpha}\big([Dy]^{-1}\big)^\beta_j\Big)
$$

where $[DY]^i_\alpha=\frac{\partial y^i}{\partial x^\alpha}$ is the Jacobian matrix for y. In particular, if the second derivatives of the transformation $x\to y(x)$ at $p\in\mathcal{M}$ are *not* all 0, then this term will have a non-zero at p. Therefore, Γ^k_{ij} does not transform under coordinate changes like a tensor field.

(b) In order to show that T is a tensor field, it suffices to show that it is $C^\infty(\mathcal{M})$-linear in its arguments; since T obviously satisfies $T(X_1+X_2,Y)=T(X_1,Y)+T(X_2,Y)$ (because ∇ and $[\cdot,\cdot]$ are $\mathbb{R}$-linear in their arguments) and $T(X,Y)=-T(Y,X)$, it suffices to show that, for any $X,Y\in\Gamma(\mathcal{M})$ and $f\in C^\infty(\mathcal{M})$:

$$
T(f\,X,Y)=f\,T(X,Y).
$$

Recall that the Lie bracket $[\cdot,\cdot]$ satisfies for any

$$
[f\,X,Y]=f[X,Y]-Y(f)\cdot X
$$

since, for any $h\in C^\infty(\mathcal{M})$:

$$
[f\,X,Y](h)=f\,X\big(Y(h)\big)-Y\big(fX(h)\big)=f\,X\big(Y(h)\big)-Y(f)X(h)-f\,Y\big(X(h)\big)=f\,[X,Y](h)-Y(f)X(h).
$$

Using the above observation and the fact that ∇ is $C^\infty(\mathcal{M})$ in its first argument and satisfies the Leibniz rule with respect to its second argument, we can calculate:

$$
\begin{aligned}
T(f\,X,Y)&=\nabla_{fX}Y-\nabla_Y(fX)-[fX,Y]\\
&=f\nabla_XY-Y(f)X-f\nabla_YX-f[X,Y]+Y(f)X\\
&=f\cdot\big(\nabla_{fX}Y-\nabla_Y(fX)-[X,Y]\big)\\
&=f\,T(X,Y).
\end{aligned}
$$

(c) As before, we have to verify that $\nabla-\bar{\nabla}$ is $C^\infty(\mathcal{M})$-linear in both its arguments; since, by the definition of a connection, both ∇ and $\bar{\nabla}$ are $C^\infty(\mathcal{M})$-linear in their first argument and $\mathbb{R}$-linear in their second argument, it remains to prove that, for any $X,Y\in\Gamma(\mathcal{M})$ and $f\in C^\infty(\mathcal{M})$:

$$
(\nabla-\bar{\nabla})(X,f\,Y)=f\,(\nabla-\bar{\nabla})(X,Y).
$$

Indeed:

$$
\begin{aligned}
(\nabla - \bar{\nabla})(X, f\,Y) &= \nabla_X(f\,Y) - \bar{\nabla}_X(f\,Y) \\
&= X(f)Y + f\,\nabla_X Y - X(f)Y - f\,\bar{\nabla}_X Y \\
&= f\,\nabla_X Y - f\,\bar{\nabla}_X Y \\
&= f\,(\nabla - \bar{\nabla})(X, Y).
\end{aligned}
$$

Therefore, setting $A(X,Y) \doteq (\nabla - \bar{\nabla})(X,Y) = \nabla_X Y - \bar{\nabla}_X Y$, we have shown that $A : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ is a $(1,2)$-tensor field; it is easy to verify that, in any local coordinate system $(x^1, \ldots, x^n)$, the components A^k_{ij} of A take the form

$$
A^k_{ij} = \Gamma^k_{ij} - \bar{\Gamma}^k_{ij}.
$$

5.2 Let $\Psi : \mathcal{M}^n \to \mathbb{R}^{n+1}$ be an immersion such that $\Psi(\mathcal{M})$ is a *spacelike* hypersurface of $(\mathbb{R}^{n+1}, \eta)$ and let $\bar{g} = \Psi_* \eta$ be the induced metric. Let $(x^1, \ldots, x^n)$ be a local coordinate chart on $\mathcal{M}$. Compute the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection associated to $\bar{g}$ in the $(x^1, \ldots, x^n)$ coordinates as functions of Ψ and its derivatives.

Solution. Let $(x^1, \ldots, x^n)$ be a local coordinate chart on $\mathcal{M}$. The components $\bar{g}_{ij}$ of the induced metric $\bar{g}$ on this chart takes the form

$$
\bar{g}_{ij} = \bar{g}\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big) = \eta\Big(\Psi^*\big(\frac{\partial}{\partial x^i}\big), \Psi^*\big(\frac{\partial}{\partial x^j}\big)\Big) = \eta_{\alpha\beta} \frac{\partial \Psi^\alpha}{\partial x^i} \frac{\partial \Psi^\beta}{\partial x^j}.
$$

We can then compute

$$
\frac{\partial \bar{g}_{jk}}{\partial x^i} = \eta_{\alpha\beta} \frac{\partial^2 \Psi^\alpha}{\partial x^i \partial x^j} \frac{\partial \Psi^\beta}{\partial x^k} + \eta_{\alpha\beta} \frac{\partial \Psi^\alpha}{\partial x^j} \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^k}.
$$

Since we assumed that $\Psi(\mathcal{M})$ is spacelike, the induced metric $\bar{g}$ is *Riemannian.* In particular, the matrix $[\bar{g}_{ij}]$ is invertible; denoting with $\bar{g}^{ij}$ the components of the inverse matrix of $\bar{g}_{ij}$, we can readily compute

$$
\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2} \bar{g}^{kl} \big(\partial_i \bar{g}_{lj} + \partial_j \bar{g}_{li} - \partial_l \bar{g}_{ij}\big) \\
&= \frac{1}{2} \bar{g}^{kl} \eta_{\alpha\beta} \Big(\frac{\partial^2 \Psi^\alpha}{\partial x^i \partial x^l} \frac{\partial \Psi^\beta}{\partial x^j} + \frac{\partial \Psi^\alpha}{\partial x^l} \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j} + \frac{\partial^2 \Psi^\alpha}{\partial x^j \partial x^l} \frac{\partial \Psi^\beta}{\partial x^i} + \frac{\partial \Psi^\alpha}{\partial x^l} \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j} \\
&\qquad\qquad - \frac{\partial^2 \Psi^\alpha}{\partial x^i \partial x^l} \frac{\partial \Psi^\beta}{\partial x^j} - \frac{\partial \Psi^\alpha}{\partial x^i} \frac{\partial^2 \Psi^\beta}{\partial x^j \partial x^l} \Big) \\
&= \bar{g}^{kl} \eta_{\alpha\beta} \frac{\partial \Psi^\alpha}{\partial x^l} \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j}
\end{aligned}
$$

(where, in passing to the last line above, we used the fact that $\eta_{\alpha\beta}$ is symmetric in α, β to write $\eta_{\alpha\beta} \frac{\partial^2 \Psi^\alpha}{\partial x^j \partial x^l} \frac{\partial \Psi^\beta}{\partial x^i} = \eta_{\alpha\beta} \frac{\partial \Psi^\alpha}{\partial x^i} \frac{\partial^2 \Psi^\beta}{\partial x^j \partial x^l}$).

Remark. An alternative way to view the above formula is as follows: Since $\Psi(\mathcal{M})$ is a spacelike hypersurface of $(\mathbb{R}^{n+1}, \eta)$ of dimension n, for any $p \in \mathcal{M}$ the tangent space $T_{\Psi(p)}\Psi(\mathcal{M})$ (which is simply the image of $d\Psi : T_p\mathcal{M} \to T_{\Psi(p)}\mathbb{R}^{n+1}$) is a spacelike hyperplane of $T_{\Psi(p)}\mathbb{R}^{n+1}$ of codimension 1. Let us denote with $\Pi_p^\top : T_{\Psi(p)}\mathbb{R}^{n+1} \to T_p\mathcal{M}$ the composition of the orthogonal projection (with respect to η) $T_{\Psi(p)}\mathbb{R}^{n+1} \to T_{\Psi(p)}\Psi(\mathcal{M})$ with the linear isomorphism $(d\Psi)^{-1} : T_{\Psi(p)}\Psi(\mathcal{M}) \to T_p\mathcal{M}$. Then, it is easy to verify that the map $\Pi_p^\top$ expressed with respect the Cartesian frame $\{e_\alpha\}_{\alpha=0}^n$ on $T_{\Psi(p)}\mathbb{R}^{n+1}$ and the $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ frame on $T_p\mathcal{M}$ takes the form

$$(\Pi_p^\top)_\alpha^k = \bar{g}^{kl}|_p \cdot \eta_{\alpha\beta} \cdot \frac{\partial \Psi^\beta}{\partial x^l}(p)$$

(you should be able to verify this by noting that $\Pi_p^\top$ maps $\Psi_*(\partial_i) = \frac{\partial \Psi^\alpha}{\partial x^i} e_\alpha$ to ∂_i and any vector η-orthogonal to $T_{\Psi(p)}\Psi(\mathcal{M}) = \mathrm{span}\{\Psi^*(\partial_i)\}_{i=1}^n$ is mapped to 0). Then, the above expression for the Christoffel symbols of $\bar{g}$ can be reexpressed as

$$\Gamma_{ij}^k = (\Pi^\top)_\beta^k \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j}.$$

Note that, since $\Psi_*(\partial_i) = \frac{\partial \Psi^\alpha}{\partial x^i} e_\alpha$, the term $\frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j}$ is simply the α-th component of $\nabla^{(\eta)}_{\Psi_*(\partial_i)} \Psi_*(\partial_j)$, where $\nabla^{(\eta)}$ is the flat connection on $\mathbb{R}^{n+1}$. Thus, the induced connection on $\mathcal{M}$ via Ψ is just the *orthogonal projection* onto $\Psi(\mathcal{M})$ of the flat connection on $\mathbb{R}^{n+1}$.

5.3 Let $\mathcal{M}$ be a smooth manifold equipped with a connection ∇. We can extend the connection ∇ to a map $\nabla : \Gamma(M) \times Ten_l^k(\mathcal{M}) \to Ten_l^k(\mathcal{M})$ by the requirements that

- ∇ satisfies the Leibniz rule with respect to tensor products, i.e. for all $X \in \Gamma(M)$

$$\nabla_X(f \otimes g) = \nabla_X f \otimes g + f \otimes \nabla_X g,$$

- ∇ commutes with contractions, i.e.

$$\nabla_X(\mathrm{tr} A) = \mathrm{tr}\big(\nabla_X A\big).$$

Show that, in any local coordinate chart $(x^1, \dots, x^n)$, if Γ_{ij}^k are the Christoffel symbols of ∇ then, for every 1-form ω:

$$(\nabla_{\frac{\partial}{\partial x^i}} \omega)_j = \partial_i \omega_j - \Gamma_{ij}^k \omega_k.$$

Moreover, for any (k,l)-tensor field T:

$$(\nabla_{\frac{\partial}{\partial x^a}} T)^{i_1 \dots i_k}{}_{j_1 \dots j_l} = \partial_a T^{i_1 \dots i_k}{}_{j_1 \dots j_l} + \Gamma_{ab}^{i_1} T^{b i_2 \dots i_k}{}_{j_1 \dots j_l} + \dots + \Gamma_{ab}^{i_k} T^{i_1 \dots i_{k-1} b}{}_{j_1 \dots j_l}$$
$$- \Gamma_{a j_1}^b T^{b i_2 \dots i_k}{}_{b j_2 \dots j_l} - \dots - \Gamma_{a j_l}^b T^{i_1 \dots i_{k-1} b}{}_{j_1 \dots j_{l-1} b}.$$

Solution. We will start by observing that, for any 1-form ω and any vector field X on $\mathcal{M}$, the function $\omega(X) \in C^\infty(\mathcal{M})$ can be seen as the contraction $\mathrm{tr}(\omega \otimes X)$ of the $(1,1)$-tensor field $\omega \otimes X$; this can be seen clearly in local coordinates, since

$$(\omega \otimes X)^i_j \doteq \omega_i X^j \quad \text{and} \quad \omega(X) = \omega_k X^k.$$

Therefore, using our assumptions that $\nabla_X(f \otimes h) = \nabla_X f \otimes h + f \otimes \nabla_X h$ and ∇ commutes with contractions, we obtain for any $X, Y \in \Gamma(\mathcal{M})$:

$$\begin{aligned} Y\big(\omega(X)\big) &= Y\big(\mathrm{tr}(\omega \otimes X)\big) \\ &= \mathrm{tr}\big(\nabla_Y(\omega \otimes X)\big) \\ &= \mathrm{tr}\big(\nabla_Y \omega \otimes X + \omega \otimes \nabla_Y X\big) \\ &= \nabla_Y \omega(X) + \omega\big(\nabla_Y X\big). \end{aligned}$$

By rearranging the terms in the above identity, we thus obtain:

$$\nabla_Y \omega(X) = Y\big(\omega(X)\big) - \omega\big(\nabla_Y X\big).$$

In any given local coordinate system $(x^1, \dots, x^n)$ on $\mathcal{M}$, if we apply the above formula for $X = \frac{\partial}{\partial x^j}$ and $Y = \frac{\partial}{\partial x^i}$ we obtain:

$$\begin{aligned} \big(\nabla_{\frac{\partial}{\partial x^i}} \omega\big)_j &= \partial_i(\omega_j) - \big(\nabla_{\partial_i}\partial_j\big)^k \omega_k \\ &= \partial_i(\omega_j) - \Gamma^k_{ij}\omega_k. \end{aligned}$$

In particular, if $\omega = dx^k$ is a coordinate 1-form, then

$$\nabla_{\partial_i}(dx^k) = -\Gamma^k_{ij} dx^j.$$

If T is a tensor field of type (k,l), then it can be expressed in a local coordinate system $(x^1, \dots, x^n)$ as before as a linear combination of the coordinate (k,l)-tensor fields $\frac{\partial}{\partial x^{\gamma_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{\delta_1} \otimes \cdots \otimes dx^{\delta_l}$, $\gamma_1, \dots, \gamma_k, \delta_1, \dots, \delta_l \in \{1, \dots, n\}$:

$$T = T^{\gamma_1 \dots \gamma_k}{}_{\delta_1 \dots \delta_l} \frac{\partial}{\partial x^{\gamma_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \cdots \otimes dx^{\delta_l}. \tag{2}$$

Our assumption on the behaviour of ∇ on tensor products and the fact that ∇ satisfies the Leibniz rule implies that, for any $f \in C^\infty(\mathcal{M})$, any $X \in \Gamma(\mathcal{M})$ and any $(Y_{(1)}, \dots, Y_{(1)}, \omega_{(1)}, \dots, \omega_{(l)}) \in \Gamma(\mathcal{M}) \times \cdots \times \Gamma(\mathcal{M}) \times \Gamma^*(\mathcal{M}) \times \cdots \times \Gamma^*(\mathcal{M})$, we have

$$\begin{aligned} \nabla_X\big(f\, Y_{(1)} \otimes \cdots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \cdots \otimes \omega_{(l)}\big) = X(f) Y_{(1)} &\otimes \cdots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \cdots \otimes \omega_{(l)} \\ &+ f\,(\nabla_X Y_{(1)}) \otimes \cdots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \cdots \otimes \omega_{(l)} \\ &+ \ldots + f\, Y_{(1)} \otimes \cdots \otimes (\nabla_X Y_{(k)}) \otimes \omega_{(1)} \otimes \cdots \otimes \omega_{(l)} \\ &+ f\, Y_{(1)} \otimes \cdots \otimes Y_{(k)} \otimes \nabla_X(\omega_{(1)}) \otimes \cdots \otimes \omega_{(l)} \\ &+ \ldots + f\, Y_{(1)} \otimes \cdots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \cdots \otimes (\nabla_X \omega_{(l)}). \end{aligned}$$

Therefore, applying this formula for the $\nabla_{\frac{\partial}{\partial x^\alpha}}$ derivative of the expression (2) and using the fact that

$$\nabla_{\partial_\alpha}\frac{\partial}{\partial x^i} = \Gamma^j_{\alpha i}\frac{\partial}{\partial x^j}, \quad \nabla_{\partial_\alpha}(dx^i) = -\Gamma^i_{\alpha j}dx^j$$

(the last formula following from our computation of the expression of ∇ acting on 1-forms), we obtain:

$$\begin{aligned}
\nabla_{\partial_\alpha}T = {} & (\partial_\alpha T^{\gamma_1\dots\gamma_k}{}_{\delta_1\dots\delta_l})\frac{\partial}{\partial x^{\gamma_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{\gamma_k}}\otimes dx^{\delta_1}\otimes\dots\otimes dx^{\delta_l} \\
& + T^{\gamma_1\dots\gamma_k}{}_{\delta_1\dots\delta_l}\,\Gamma^\beta_{\alpha\gamma_1}\frac{\partial}{\partial x^\beta}\otimes\dots\otimes\frac{\partial}{\partial x^{\gamma_k}}\otimes dx^{\delta_1}\otimes\dots\otimes dx^{\delta_l} \\
& + \ldots + T^{\gamma_1\dots\gamma_k}{}_{\delta_1\dots\delta_l}\,\Gamma^\beta_{\alpha\gamma_k}\frac{\partial}{\partial x^{\gamma_1}}\otimes\dots\otimes\frac{\partial}{\partial x^\beta}\otimes dx^{\delta_1}\otimes\dots\otimes dx^{\delta_l} \\
& - T^{\gamma_1\dots\gamma_k}{}_{\delta_1\dots\delta_l}\,\Gamma^{\delta_1}_{\alpha\beta}\frac{\partial}{\partial x^{\gamma_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{\gamma_k}}\otimes dx^\beta\otimes\dots\otimes dx^{\delta_l} \\
& - \ldots - T^{\gamma_1\dots\gamma_k}{}_{\delta_1\dots\delta_l}\,\Gamma^{\delta_l}_{\alpha\beta}\frac{\partial}{\partial x^{\gamma_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{\gamma_k}}\otimes dx^{\delta_1}\otimes\dots\otimes dx^\beta.
\end{aligned}$$

Therefore, considering the $\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_k}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_l}$ component of the above expression (noticing that, in each summand involving Γ, an index of Γ is contracted with one index of T, and we are free to rename those indices as we please), we obtain

$$\begin{aligned}
(\nabla_{\frac{\partial}{\partial x^a}}T)^{i_1\dots i_k}{}_{j_1\dots j_l} = \partial_a T^{i_1\dots i_k}{}_{j_1\dots j_l} & + \Gamma^{i_1}_{ab}T^{bi_2\dots i_k}{}_{j_1\dots j_l} + \ldots + \Gamma^{i_k}_{ab}T^{i_1\dots i_{k-1}b}{}_{j_1\dots j_l} \\
& - \Gamma^b_{aj_1}T^{i_1 i_2\dots i_k}{}_{bj_2\dots j_l} - \ldots - \Gamma^b_{aj_l}T^{i_1\dots i_k}{}_{j_1\dots j_{l-1}b}.
\end{aligned}$$

5.4 Let $(\overline{\mathcal{M}},\bar g)$ be a *Riemannian* manifold (i.e. $\bar g$ is positive definite) and let us define the Lorentzian manifold $(\mathcal{M},g)$ so that $\mathcal{M}=\mathbb{R}\times\overline{\mathcal{M}}$ and g is the product metric $g=-(dt)^2+\bar g$; this means that, for every local coordinate chart $(x^1,\dots,x^n)$ on $\mathcal{U}\subset\overline{\mathcal{M}}$, if we extend it to a local coordinate chart $(t,x^1,\dots,x^n)$ on $\mathbb{R}\times\mathcal{U}\subset\mathcal{M}$ so that t is simply the projection on the $\mathbb{R}$ factor, then

$$g = -dt^2 + \bar g_{ij}dx^i dx^j.$$

Show that a curve $\gamma:(0,1)\to(M,g)$ is a geodesic (for the Levi-Civita connection of g) if and only, in any local coordinate system $(t;x^1,\dots,x^n)$ as above, if it can be written in the form

$$\gamma(s) = (t(s);\bar\gamma^i(s))$$

where $t(s)=\lambda_1 s+\lambda_0$ for some $\lambda_1,\lambda_0\in\mathbb{R}$ and $\bar\gamma:(0,1)\to\overline{\mathcal{M}}$ is a geodesic of $(\overline{\mathcal{M}},\bar g)$.

Solution. For any $p\in\mathcal{M}$, let $(x^0,x^1,\dots,x^n)=(t,x^1,\dots,x^n)$ be a local coordinate system around p which is as described in the statement of the exercise. We will adopt the following convention: We will use Greek letters (i.e. $\alpha,\beta,\gamma,\dots$) for indices ranging from 0 to n and Latin letters (i.e. $i,j,k,\dots$) for indices ranging from 1 to n. With this notation, the components $g_{\alpha\beta}$ of g take the form

$$g_{00}=-1, \quad g_{0i}=0, \quad g_{ij}=\bar g_{ij}.$$

Therefore, we can also calculate that the components $g^{\alpha\beta}$ of the inverse matrix of $[g]$ take the form

$$g^{00} = -1, \quad g^{0i} = 0, \quad g^{ij} = \bar{g}^{ij}$$

(where $\bar{g}^{ij}$ are the components of the inverse matrix of $[\bar{g}_{ij}]$).

Let us, now turn to calculating the Christoffel symbols $\Gamma^{\delta}_{\alpha\beta}$ of g. Using the formula

$$\Gamma^{\delta}_{\alpha\beta} = \frac{1}{2} g^{\delta\lambda}\big(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}\big),$$

we can readily verify that

$$\Gamma^{\delta}_{\alpha\beta} = 0 \quad \text{when at least one of } \alpha, \beta, \delta \text{ is } 0$$

and

$$\Gamma^{k}_{ij} = \bar{\Gamma}^{k}_{ij} \quad \text{when } i, j, k \in \{1, \dots, n\},$$

where $\bar{\Gamma}^{k}_{ij}$ are the Christoffel symbols of $\bar{g}$.

Let $\gamma : (0,1) \to \mathcal{M}$, $\gamma(s) = \big(x^0(s), x^1(s), \dots, x^n(s)\big)$ be a geodesic of g. Thus, the components of γ satisfy the geodesic ODE

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\delta}\dot{x}^{\beta}\dot{x}^{\delta} = 0.$$

Applying the above relation for $\alpha = 0$ and using the fact that (as we calculated) $\Gamma^{0}_{\beta\gamma} = 0$, we obtain

$$\ddot{x}^0(s) = 0 \Rightarrow x^0(s) = \lambda_1 s + \lambda_0 \quad \text{for some } \lambda_0, \lambda_1 \in \mathbb{R}.$$

Similarly, applying the above relation for $\alpha = k \in \{1, \dots, n\}$, we obtain

$$\begin{aligned} 0 &= \ddot{x}^k + \Gamma^{k}_{\beta\delta}\dot{x}^{\beta}\dot{x}^{\delta} \\ &= \ddot{x}^k + \Gamma^{k}_{ij}\dot{x}^{i}\dot{x}^{j} + \Gamma^{k}_{i0}\dot{x}^{i}\dot{x}^{0} + \Gamma^{k}_{0j}\dot{x}^{0}\dot{x}^{j} + \Gamma^{k}_{00}\dot{x}^{0}\dot{x}^{0} \\ &= \ddot{x}^k + \bar{\Gamma}^{k}_{ij}\dot{x}^{i}\dot{x}^{j} + 0, \end{aligned}$$

i.e. the curve $s \to \big(x^1(s), \dots, x^n(s)\big)$ satisfies the geodesic equation with respect to the metric $\bar{g}$.

Remark. The above proof can be easily generalised to the case of a pseudo-Riemannian manifold $(\mathcal{M}, g)$ which is the product of the pseudo-Riemannian manifolds $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$. In that case, the projections γ_1 and γ_2 of any geodesic γ of $(\mathcal{M}, g)$ on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, are geodesics for g_1 and g_2; the proof uses the fact that, in any product coordinate system $(x^1, \dots, x^{n_1}; y^1, \dots, y^{n_2})$ on $\mathcal{M}$ (where $(x^1, \dots, x^{n_1})$ and $(y^1, \dots, y^{n_2})$ are local coordinates on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively), any Christoffel symbol $\Gamma^{\delta}_{\alpha\beta}$ with *mixed* indices (i.e. with indeices belonging to both $(x^1, \dots, x^{n_1})$ and $(y^1, \dots, y^{n_2})$) has to vanish.

5.5 In this exercise, we will prove that there exist compact Lorentzian manifolds which are **geodesically incomplete** (recall that, as a consequence of the Hopf–Rinow theorem in Riemannian geometry, every compact Riemannian manifold is geodesically complete). Consider the manifold $\mathcal{M} = \mathbb{R}^2 \setminus 0$ equipped with the metric

$$g = \frac{1}{u^2 + v^2} du dv.$$

(a) Verify that $(\mathcal{M}, g)$ is a smooth Lorentzian manifold and that the map $(u, v) \to (\lambda \cdot u, \lambda \cdot v)$ is an isometry for every $\lambda \neq 0$.

(b) Consider the group of isometries $\Gamma = \big\{(u, v) \to (2^k u, 2^k, v),\ k \in \mathbb{Z}\big\}$. Show that the quotient space $\mathcal{M}/\Gamma$ is a compact manifold. Show also that $\mathcal{M}/\Gamma$ inherits a natural metric $\tilde{g}$ from $(\mathcal{M}, g)$ so that the quotient map $(\mathcal{M}, g) \to (\mathcal{M}/\Gamma, \tilde{g})$ is a local isometry.

(c) Show that the map $(\mathcal{M}, g) \to (\mathcal{M}/\Gamma, \tilde{g})$ maps geodesics to geodesics. Compute the geodesic equation on $(\mathcal{M}, g)$ and deduce that $(\mathcal{M}/\Gamma, \tilde{g})$ contains a geodesic $\gamma : (a, b) \to \mathcal{M}/\Gamma$ with $b < +\infty$ which *cannot* be extended beyond $t = b$.

Solution. (a) It is straightforward to verify that g is a smooth Lorentzian metric on the smooth manifold $\mathcal{M} = \mathbb{R}^2 \setminus 0$ (in fact, it is conformal to the Minkowski metric on $\mathbb{R}^2 \setminus 0$). For any $\lambda \in \mathbb{R} \setminus 0$, we can readily compute that the map $T_\lambda : \mathcal{M} \to \mathcal{M}$, defined by

$$T_\lambda(u, v) = (\lambda u, \lambda v),$$

is a diffeomorphism satisfying

$$\begin{aligned}(T_\lambda)_* g &= \frac{1}{(\lambda u)^2 + (\lambda v)^2} d(\lambda u) d(\lambda v) \\ &= \frac{1}{u^2 + v^2} dudv \\ &= g.\end{aligned}$$

Therefore, T_λ is an isometry of $(\mathcal{M}, g)$

(b) Let's recall first a few things about the quotient of a manifold by a subgroup of diffeomorphisms: Let G be a subgroup of Diff($\mathcal{N}$) for a smooth manifold $\mathcal{N}$. Setting, for any point $p \in \mathcal{N}$,

$$[p]_G \doteq \big\{q \in \mathcal{N} : q = F(p) \text{ for some } F \in G\big\},$$

then the set

$$\mathcal{N}/G \doteq \{[p]_G : p \in \mathcal{N}\}$$

(which is called *the quotient* of $\mathcal{N}$ by the action of G), equipped with the quotient topology, has the structure of a smooth manifold if and only if, for any $p \in \mathcal{N}$, there exists an open neighborhood $\mathcal{U} \subset \mathcal{N}$ of p such that

$$\mathcal{U} \cap F(\mathcal{U}) = \emptyset \quad \text{for all } \ F \in G \tag{3}$$

(it is straightforward to verify that, for any $p \in \mathcal{M}$, if $\Phi : \mathcal{V} \to \mathbb{R}^n$ is a smooth coordinate chart on a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p, then the collection of coordinate charts

$$\tilde{\Phi} = \big\{\Phi \circ F^{-1} : F \in G\big\}$$

is a G-invariant set of coordinate charts on neighborhoods of all the points in $[p]_G$ and can be used to construct a coordinate chart around $[p]_G$ in $\mathcal{N}/G$). With this manifold structure on $\mathcal{N}/G$, the quotient map $\pi : \mathcal{N} \to \mathcal{N}/G$, $p \to [p]_G$, is a local (but not global) diffeomorphism. Notice that, for any curve $\gamma : (a,b) \to \mathcal{N}/G$, the preimage of γ in $\mathcal{N}$ consists of the family of curves

$$\pi^{-1}(\gamma) = \bigcup_{F\in G} \gamma_F,$$

satisfying

$$F_1(\gamma_{F_2}) = \gamma_{F_1\circ F_2}, \quad \text{for all } F_1, F_2 \in G.$$

By considering the tangent vectors to such curves, we infer the following statement about $T(\mathcal{N}/G)$: For any $p \in \mathcal{N}$ and any tangent vector $v \in T_{[p]_G}(\mathcal{N}/G)$, there exists a family of tangent vectors $v_F \in T_{F(p)}\mathcal{N}$, $F \in G$, such that

$$\pi^*(v_F) = v \quad \text{for all } F \in G$$

and satisfying

$$F_1^*(v_{F_2}) = v_{F_1\circ F_2} \quad \text{for any } F_1, F_2 \in G.$$

Returning to our case (where $\mathcal{N} = \mathcal{M}$ and $G = \Gamma$), in order to verify that $\mathcal{M}/\Gamma$ is a *compact* manifold, it suffices to show that there exists a compact subset $\mathcal{K} \subset \mathcal{M}$ such that the quotient map π is onto when restricted to $\mathcal{K}$ (compactness of $\mathcal{M}/\Gamma$ in this case follows from the fact that, since π is continuous, $\pi(\mathcal{K})$ is necessarily compact). We can readily verify that

$$\mathcal{K} = \{(u,v) \in \mathbb{R}^2 \setminus 0 : \frac{1}{2} \leqslant u^2 + v^2 \leqslant 2\}$$

has this property (which can be equivalently reexpressed as the statement that, for every $p \in R^2 \setminus 0$, there exists an $F \in \Gamma$ such that $F(p) \in \mathcal{K}$).

We will now use the fact that Γ is in fact a group of isometries to deduce that the quotient manifold $\mathcal{M}/\Gamma$ admits a *quotient metric* $\tilde{g}$. It is natural to define, for any $[p]_G \in \mathcal{M}/\Gamma$ and any $v, w \in T_{[p]_G}\mathcal{M}/\Gamma$,

$$\tilde{g}(v,w) \doteq g(v_F, w_F) \quad \text{for all } F \in \Gamma \tag{4}$$

(see the the discussion above for the notation v_F, w_F). The above definition, of course, makes sense only when the right hand side of (4) is the same for all $F \in \Gamma$; this is true precisely when Γ is a group of isometries of $(\mathcal{M}, g)$, since then $g(v_F, w_F) = g(F^*v_1, F^*v_2)$ is equal to $g(v_1, w_1)$ (1 being the identity element in Γ). Moreover, since $\pi^*(v_F) = v$, (4) trivially implies that, in this case, the quotient map π is a local isometry.

Remark. The above argument works in the case of any group of isometries G acting on a pseudoriemannian manifold $(\mathcal{N}, g)$ in a way that (3) holds.

(c) In general, if $\Psi : (\mathcal{N}_1, g_1) \to (\mathcal{N}_2, g_2)$ is a local isometry, then, for any $X, Y \in \Gamma(\mathcal{N}_1)$, we have $\nabla^{(\mathcal{N}_2)}_{\Psi^*X}(\Psi^*Y) = \nabla^{(\mathcal{N}_1)}_X Y$ (where $\nabla^{(\mathcal{N}_i)}$ denotes the Levi-Civita connection of $(\mathcal{N}_i, g_i)$; this can be readily verified using the formula of Kozul for any vector fields U, V, W:

$$2g_i\big(\nabla^{(\mathcal{N}_i)}_U V, W\big) = U\big(g(V,W)\big) + V\big(g(U,W)\big) - W\big(g(U,V)\big) - g([V,W],U) - g([U,W],V) + g([U,V],W)$$

(using $U = X, V = Y, W = Z \in \Gamma(\mathcal{N}_1)$ for $i = 1$ and $U = \Psi^*X, V = \Psi^*Y, W = \Psi^*Z$ for $i = 2$). Thus, if $\gamma : I \to \mathcal{N}_1$ is a geodesic of g_1, i.e. satisfies $\nabla^{(\mathcal{N}_1)}_{\dot\gamma}\dot\gamma = 0$, then $\nabla^{(\mathcal{N}_2)}_{\Psi^*\dot\gamma}(\Psi^*\dot\gamma) = 0$, i.e. $\Psi(\gamma)$

is a geodesic of $(\mathcal{N}_2, g_2)$. Therefore, since, in our case, the quotient map $\pi : \mathcal{M} \to \mathcal{M}/\Gamma$ is a local isometry, it maps geodesics to geodesics.

In the (u, v) coordinate system on $\mathcal{M}$, we can readily compute that the Christoffel symbols of the Levi-Civita connection of g take the following form:

$$\Gamma^u_{uu} = -\frac{2u}{u^2+v^2}, \quad \Gamma^v_{vv} = -\frac{2v}{u^2+v^2}, \quad \Gamma^u_{uv} = \Gamma^u_{vv} = \Gamma^v_{uv} = \Gamma^v_{uu} = 0.$$

Therefore, the geodesic equation takes the following form: If $s \to (u(s), v(s))$ is a geodesic of $(\mathcal{M}, g)$, then

$$\begin{aligned} \ddot{u} - \frac{2u}{u^2+v^2}(\dot{u})^2 = 0, \\ \ddot{v} - \frac{2v}{u^2+v^2}(\dot{v})^2 = 0. \end{aligned}$$

It can be easily verified that the curve $s \to (u(s), v(s)) = (\frac{1}{s}, 0)$, $s \in (-\infty, 0)$ is a null geodesic of $(\mathcal{M}, g)$, which is maximally extended (since, as a subset of $\mathbb{R}^2$, the limit point of this curve is $(0,0)$ as $s \to 0$). The projection of this curve on $\mathcal{M}/\Gamma$ is, therefore, a maximally extended geodesic of $\tilde{g}$.